Building Up from Nothing

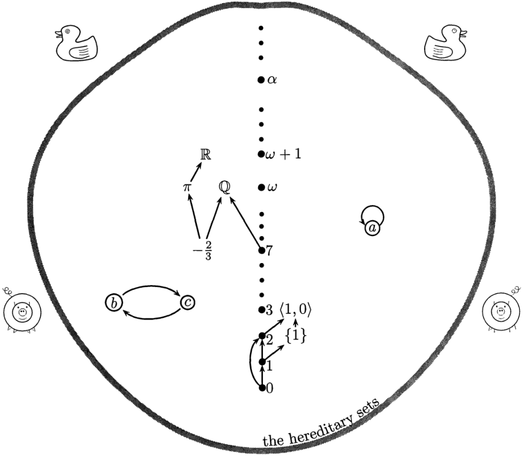
Columbia Undergraduate Math Society

June 20, 2021

Outline

Where are we?

Figure I.1: The Set-Theoretic Universe in ZF^-



Earlier

Theorem

There is at most one empty set.

Proof.

See last time. ☐

We still need to show that there is *an* empty set; after all not all sets $\{x \mid Q(x)\}$ where Q is some general property exists. Famously,

Theorem

Take $R = \{x \mid x \notin x\}$*, so we have* $R \in R \iff R \notin R$*. In particular, this suggests that there cannot be a universal set:* $\forall z \exists y (y \notin z)$*.*

Proof.

Given some universal set z, consider $R = \{x \in z \mid x \notin x\}$, such that if $R \in z$, then $R \in R \iff R \notin R$, so $R \notin z$. $\Rightarrow\!\Leftarrow$ ☐

The Empty Set

- Axiom 3 (Comprehesion): For each formula φ, without y free,

$$\exists y \forall x (x \in y \iff x \in z \wedge \varphi(x))$$

  Idea: for any set (z) and some property (φ), there is some set (y) with only elements that satisfy this property.

But what is a formula? Vaguely, an expression made with $\in, =, \wedge, \vee, \neg, \forall, \exists$, variables, etc.

Theorem

The empty set exists: $\emptyset$ will denote the (unique) set y such that $\forall x (x \notin y)$.

Proof.

Start with any set z (see Axiom 0: a set exists!) and apply Comprehension with $x \neq x$, so we get some statement like $\exists y \forall x (x \in y \iff x \in z \wedge x \neq x)$. □

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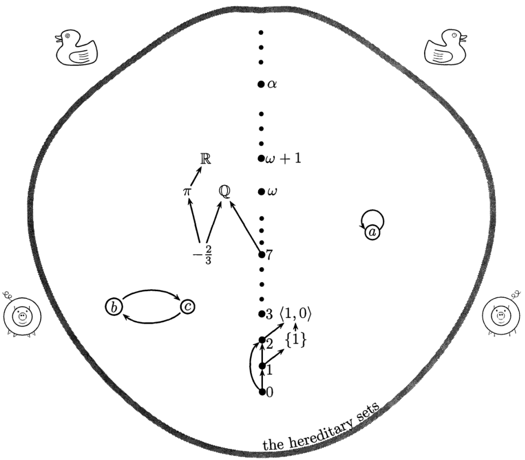

Nonempty Sets

Note that we can only make smaller sets with Comprehension:

Definition

Given sets y, z,

- $y \cap z = \{x \in y \mid x \in z\}$
- $y \setminus z = \{x \in y \mid x \notin z\}$

Naively, we still only have the empty set!

- Axiom 4 (Pairing):

$$\forall x \forall y \exists z (x \in z \land y \in z)$$

- Axiom 5 (Union):

$$\forall \mathcal{F} \exists A \forall Y \forall x \left(x \in Y \land Y \in \mathcal{F} \implies x \in A\right)$$

Idea: given a family of sets ($\mathcal{F}$), we can flatten it into a "single" set as the union of all its members.

Unions and Intersections

Definition

$$\bigcup \mathcal{F} = \bigcup_{Y \in \mathcal{F}} Y = \{x \mid \exists Y \in \mathcal{F}(x \in Y)\}$$

$$\bigcap \mathcal{F} = \bigcap_{Y \in \mathcal{F}} Y = \{x \mid \forall Y \in \mathcal{F}(x \in Y)\}$$

Note that intersection is already strictly defined from Comprehension.

For union, we take A as in the Union axiom, and apply Comprehension with the formula above.

For intersection, we need that $\mathcal{F} \neq \emptyset$. Why?

Ordinals?

Example

Take $x = y = \emptyset$; then we get by Pairing (and Comprehension) that $\{\emptyset\}$ exists. Then, by Pairing, $\{\emptyset, \{\emptyset\}\}$ also exists.

Definition

The ordinal successor function is $S(x) = x \cup \{x\}$, so we can get

$$\begin{aligned} 1 &= S(0) = \{0\} \\ 2 &= S(1) = \{0, 1\} \\ 3 &= S(2) = \{0, 1, 2\} \end{aligned}$$

etc.

So we can finally get beyond just the empty set! (More on ordinals later - likely next week).

Where are we?

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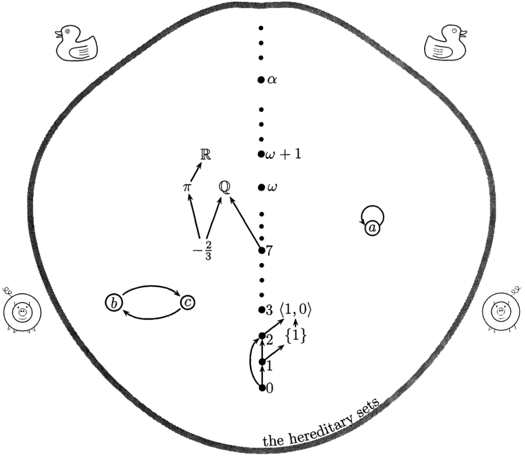


Relations

Definition

A binary relation is a set of ordered pairs; that is, R is a binary relation when

$$\forall u \in R \exists x, y (u = \langle x, y \rangle)$$

and we abbreviate $\langle x, y \rangle \in R$ to xRy; similarly, $\langle x, y \rangle \notin R$ becomes $x \not R y$.

- R is transitive on A if $\forall x, y, z \in A\,(xRy \wedge yRz \implies xRz)$.
- R is reflexive on A if $\forall x \in A(xRx)$.
- R is irreflexive on A if $\forall x \in A(x \not R x)$.
- R satisfies trichotomy on A if $\forall xy \in A(xRy \vee yRx \vee x = y)$.
- R is symmetric on A if $\forall xy \in A(xRy \iff yRx)$.

More Relations

- Strict partial orders are irreflexive, transitive relations.
- Strict total orders are irreflexive, transitive relations which satisfy trichonomy.
- Equivalence relations are reflexive, symmetric, and transitive relations.

Examples

Consider $\mathbb{Z}$: then, $<$ is a strict total order; $\leq$ is not. For a partial order, consider $\mathcal{P}(\{x, y\})$ equipped with $\subset$.

Note that being irreflexive and transitive is enough to show $xRy \implies y\not{R}x$.

A Quick Detour: Functions

Definition

For any set R, define

$$\text{dom}(R) = \{x \mid \exists y(\langle x, y\rangle \in R)\}$$
$$\text{ran}(R) = \{y \mid \exists x(\langle x, y\rangle \in R)\}$$

Relations are not functions; functions are relations:

Definition

A relation R is a function if for every $x \in \text{dom}(R), \exists! y$ such that $\langle x, y\rangle \in R$; in particular, we put $R(x)$ to denote that y.

Domain and Range

As a reminder that this series is still about set theory: how do we justify the construction of domain and range?

Definition

For any set R, define

$$\mathrm{dom}(R) = \{x \mid \exists y(\langle x, y\rangle \in R)\}$$
$$\mathrm{ran}(R) = \{y \mid \exists x(\langle x, y\rangle \in R)\}$$

Remember that $\langle x, y\rangle = \{\{x\}, \{x, y\}\}$; then we have that $\{x\}, \{x, y\} \in \bigcup R$, and as a result $x, y \in \bigcup\bigcup R$; Comprehension immediately gives the definition of domain (and range):

$$\mathrm{dom}(R) = \{x \in \bigcup\bigcup R \mid \exists y(\langle x, y\rangle \in R)\}$$

More Sets!

A lot of times one might see a function $S \to T$ as a subset of $S \times T$; *ab initio*, we don't even know $S \times T$ exists!

- Axiom 6 (Replacement): For each formula φ, without B free,

  $$\forall x \in A \exists! y \varphi(x, y) \to \exists B \forall x \in A \exists y \in B \varphi(x, y)$$

  Idea: given a set (A) and some other class of sets (y) associated to elements of the first set (x), the latter can be formed into a set (B).

This lets us create sets in the form $S = \{y \in B \mid \exists x \in A \varphi(x, y)\}$.

Example

$$S \times T = \{x \mid \exists s \in S \exists t \in T (x = \langle s, t \rangle)\}$$

Isomorphisms

Theorem

Suppose $\forall x \in A \exists ! y \varphi(x, y)$. *Then there is a function* f *with domain* A *taking* x *to that associated* y.

Proof.

Replacement axiom. □

Definition

F is an isomorphism from $(A; <)$ to $(B; \triangleleft)$ if F is a bijection $A \to B$ and

$$\forall x, y \in A (x < y \iff F(x) \triangleleft F(y))$$

This codifies formally when two (ordered) sets are structurally the same, even if the sets they are composed of aren't equal as sets.

Well Orderings

Definition

Let R be a relation. $y \in X$ is R-minimal in X if

$$\neg \exists z (z \in X \wedge zRy)$$

and R-maximal in X if

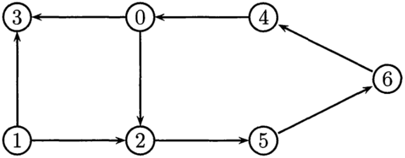
$$\neg \exists z (z \in X \wedge yRz).$$

Further, R is well-founded on A if for all non-empty $X \subset A$, there is some $y \in X$ which is R-minimal on X.

Example

0 is $<$-minimal in $\mathbb{N}$, and $<$ is well founded. (Remember that $\mathbb{N}$ doesn't really exist yet!)

A Better Example



For example, $0R2$ in this diagram.

What are the R-minimal elements on $7 = \{0, 1, 2, 3, 4, 5, 6\}$? R-maximal? Is R well-founded on 7? (Think about what a cycle in this graph means!)

Well Orderings

Definition

R well-orders A if R is a strict total order on A and is also well-founded on A.

In particular, this suggests that any subset of A can only have a singular least element (via trichotomy). This leads us to the equivalent formulation that a well-order is a strict total order where every (non-empty) subset has a least element.

Ordinals

Definition

z is a transitive set if $\forall y \in z(y \subseteq z)$; equivalently,

$$\forall x, y(x \in y \wedge y \in z \implies x \in z)$$

Definition

z is a (von Neumann) ordinal if z is a transitive set and z is well-ordered by $\in$.

Remember: the ordinal successor function is $S(x) = x \cup \{x\}$, so

$$1 = S(0) = \{0\}$$
$$2 = S(1) = \{0, 1\}$$
$$3 = S(2) = \{0, 1, 2\}$$
$$\vdots$$

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